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Elliptical Sundials: General & Craticular

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The analemmatic dial, that indicates time by the azimuth of a vertical stile's shadow on an elliptical dial plate, was first described in 1640 in a short pamphlet by the French mathematician Vaulezard. Four years later he produced a longer publication that included the first proof of the analemmatic dial. The whole thing amounted to 42 pages. In the present article, I am more interested in presenting to you some of the slightly later work on this sort of elliptical dial by Samuel Foster who was professor of astronomy at Gresham College in London.

Shortly after his death in 1652, Foster's book *Elliptical or Azimuthal Horologiography* appeared in print - 204 pages – promising to cover much more than Vaulezard's short works: *'Tis true, that Mr. Vaulezard… sheweth by the projection of an Ellipsis upon the Plain of the Horizon, and by the help of an upright moveable stile to finde the Houre and Azimuth…. But this Treatise of our Author is very different from that, and most of the things here handled … are not appliable to his, and [are] in themselves wholly new.* Bold claims appeared in the prefaces of many 17th century books – even the most pedestrian of them. But does this book follow through on its promise?

Consider this very interesting statement. *[R]epresenting the true Hours by the shadow made by the Axis of the World is but one of those infinite wayes which may be invented….* This is interesting for two reasons. It appears to hold out the promise of relating elliptical sundials – what we might still be thinking of as simply analemmatics – to the traditional polar axis sundials that seem to be of a very different type. And we have this promise - while most dialists are mired in finding yet another way to lay out the same old hourlines, that Foster is offering an infinite number of new ways to show the hours.

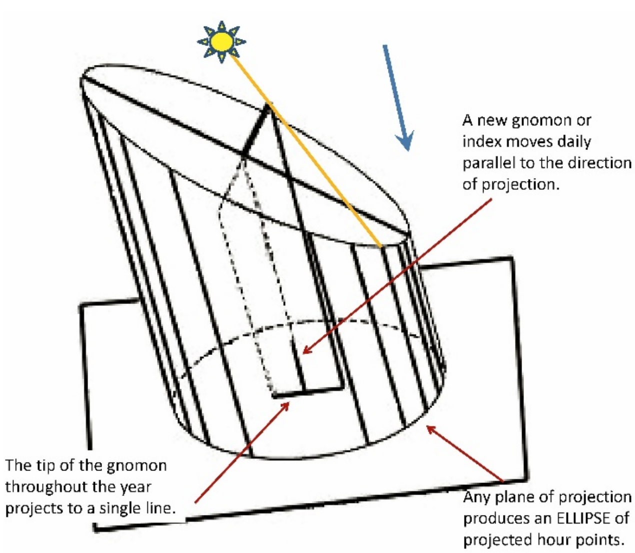


Fig. 1

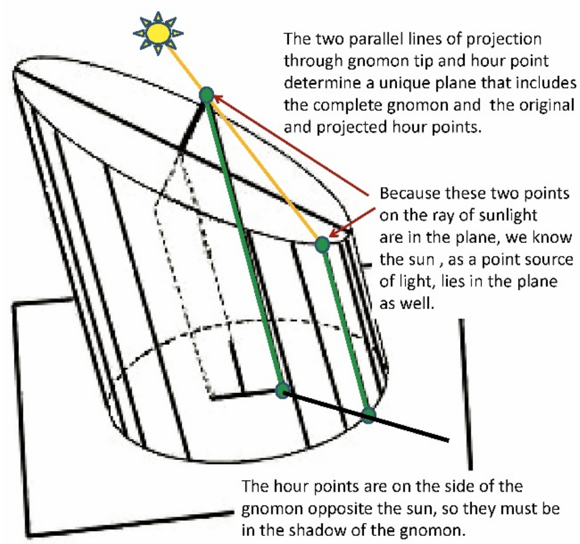


Fig. 2

Let's begin by considering a Circular Equatorial sundial (Figs. 1-2). Let the gnomon on this dial have a length g that varies from day to day, according to this equation: $g = r\tan\delta$ (r being the radius of the dial face and δ the solar declination of the day. This dial has an important interesting feature.

Not only does the gnomon's shadow indicate the correct hour point, but at ALL TIMES during the day, the shadow of the TIP OF THE GNOMON falls on the circumference of the dial. Now select ANY direction of projection and project the dial and hour points onto any planar surface. Any plane of projection produces an ELLIPSE of projected hour points. The tip of the gnomon throughout the year projects to a single line. A new gnomon or index moves daily, parallel to the direction of projection.

Obviously, in the special case of a vertical projection, we have the makings of an analemmatic sundial. But we have not restricted the construction to a vertical projection. We need to prove that what we have will serve as a sundial. So Foster is able to prove this very general result: Given any plane and an index pointing in any direction at any angle (but movable parallel to itself), it is possible to create a sundial using only an ellipse and a straight line.

This result is not so surprising today – I have even seen a very similar proof given in a paragraph or two on the Sundial Mailing List. But, remember, Foster is living in the 17th century, a time at which the result was far from obvious (it may not be that obvious even today).

When I was in graduate school and first found Foster's work, I searched and searched to find anyone else who might have shared in this insight. In the days before the Internet and Google, all I could find was a 1902 manuscript by Louis Gruey, the head of the Observatory in Besançon France. Gruey goes through many pages and uses calculus to get a similar result – but his findings are restricted to gnomons in the meridian plane only. So he has to use calculus – that hadn't even been invented in Foster's time – and he only gets half the result. His purpose was simply to develop a proof for the dial in Besançon – which is now the world's 3rd oldest extant monumental analemmatic dial – recently completely restored.

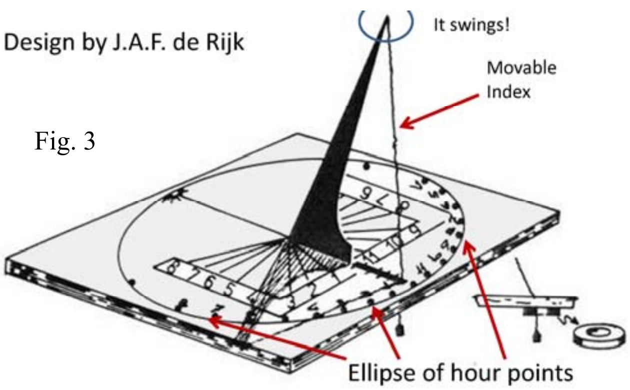


Fig. 3

In his article *Equator Projection Sundials* (*Journal of the British Astronomical Association, 1986)*, Hans de Rijk gives a modern analysis that begins by acknowledging the pioneering work of Samuel Foster – although it is not clear from the text that he had actually read Foster's book in full. But he does reproduce Foster's results and he quite nicely extends analemmatic dials to get a whole new breed of sundial. Instead of applying a Parallel projection to all the hour points and the gnomon tips, de Rijk considers applying a nonparallel projection from a single point. See Fig. 3 for an example of his resulting dial. It is similar to an analemmatic in that it has an ellipse of hour points and a movable index or stile, but one end of the stile remains stationary, so the stile does not move parallel to itself with changing solar declination. Instead, it moves so that one end is held constant while the other moves. Yvon Massé published a similar study *Central Projection Analemmatic Sundials, The Compendium* 5(1):4-9, Mar 1999), in which he showed that with a central projection, you can have hour points arranged in elliptical, parabolic or hyperbolic arrangements.

But this goes well beyond Foster. I want to return to examine the brilliant work he did.

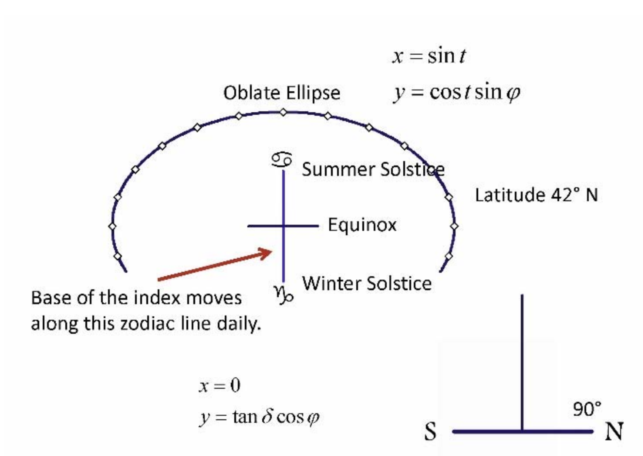


Fig. 4 Analemmatic Sundial

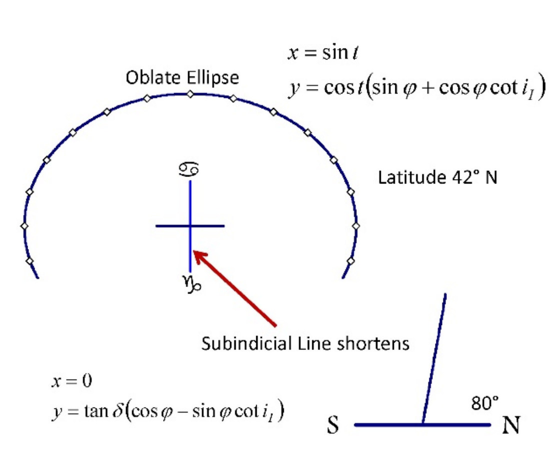


Fig. 5 Elliptical Sundial – Inclining Index

Begin with an analemmatic (Fig. 4). This is the familiar configuration of hour points, zodiac and index – complete with the relevant equations. Note that the ellipse is oblate – its horizontal axis is larger than the vertical. Now let's change the inclination of the index so that it is no longer vertical (Fig. 5). If it remains in the meridian plane but now has an inclination to the north of, say, 80˚, the ellipse changes slightly, and the subindicial line gets shorter. The equations have changed – but that's necessary to take the *inclination into account.*

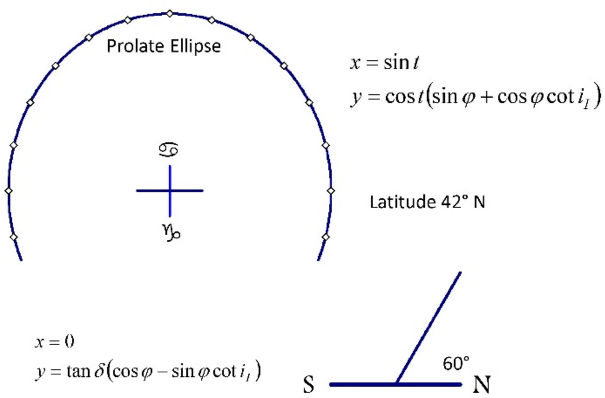


Fig. 6 Elliptic Dial with Prolate Ellipse

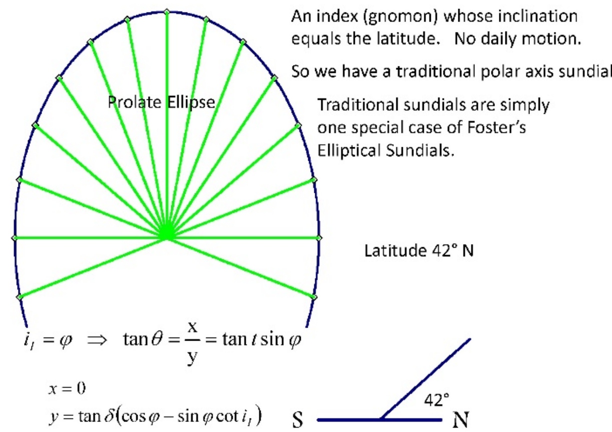


Fig. 7 Elliptic Dial with Index Parallel to Polar Axis

At some point, as we continue to incline the index, the ellipse becomes prolate (Fig. 5) – its vertical axis becomes longer than its horizontal. Finally, when the inclination of the index equals the latitude (Fig. 7), the length of the zodiac (or declination scale) is 0. What that means is that the index, now parallel to the polar axis, does not move from day to day – it is stationary. If we do the math, we find that when the inclination equals the latitude, the angles to the hour points are simply the angles for a traditional sundial. We can draw in hourlines and thus have a traditional polar axis dial. The traditional dial is thus simply a special case of Foster's elliptical sundials.

If we were to continue to decrease the inclination of the index, the zodiac would come back, with positive and negative declinations now reversed! Instead, let us look at the shift from Oblate to Prolate ellipses (Fig. 8). Somewhere in our process of changing inclination, the ellipse became an exact circle. Foster showed that the circle results when the index bisects the angle between the polar axis and the axis of the dial plane. With a circle, we get equispaced hour points. See Fig. 9 for Foster's version of the resulting dial – the first Foster circular dial.

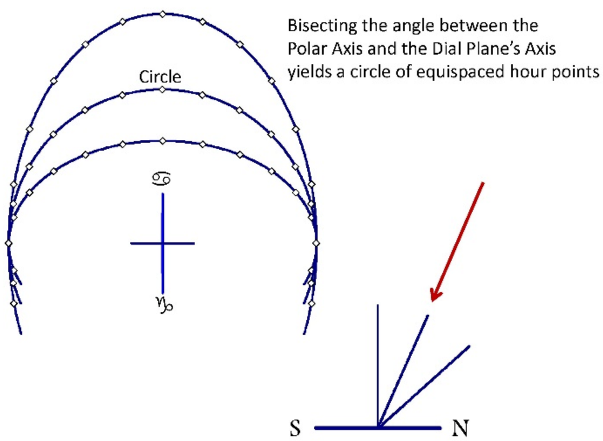


Fig. 8 Transition from Oblate to Prolate Ellipse

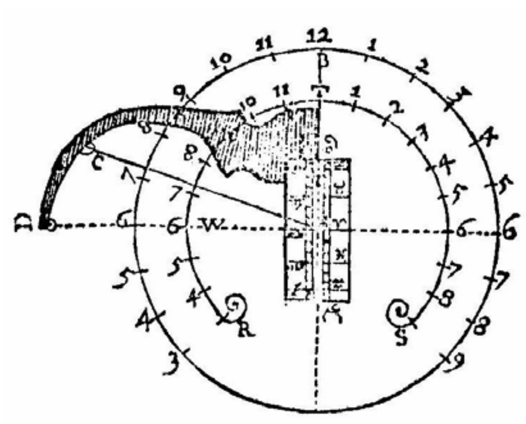


Fig. 9 The first image of a Foster Circular Dial

The discussion of this interesting new dial filled an entire section of Foster's book. Unfortunately, the discovery seems to have been lost for many years. 121 years later, we have a separate independent discovery in Germany. The brilliant mathematician J.H. Lambert introduced 'his discovery' of a new kind of sundial in 1775 – which was simply Foster's Circular dial (*Eine neue Art Sonnenuhren*, Astronomisches Jahrbuch oder Ephemeriden für das Jahr 1777, Berlin, 1775). But this discovery, too, seems to have been quickly forgotten.

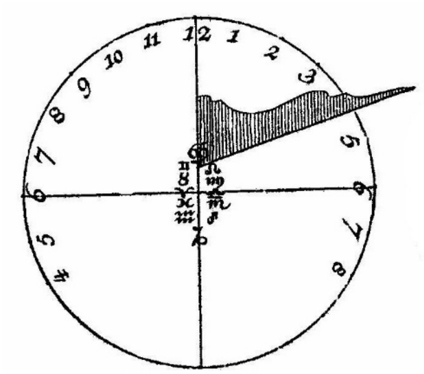


Fig. 10 Lambert's 1775 version of the dial

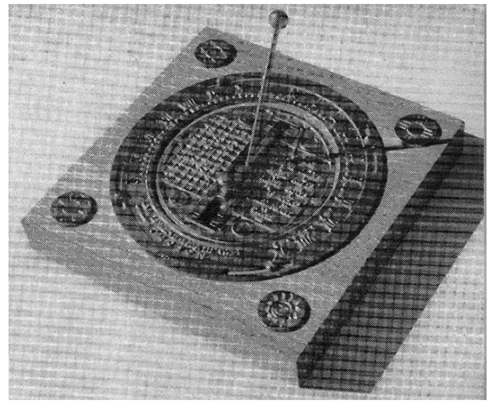


Fig. 11 Ericson's 1972 version of the circular dial

Fig. 12 The Greenwich Observatory Tercentenary Sundial – at Herstmonceux Castle

Two centuries later, in 1972 it was 'discovered' again by Albert Ericson (*A Hybrid Sundial*, Sky & Telescope, 1972, 44(11):299,305). It was Ericson's article that launched me on my investigation of elliptical dials and eventually led me back in time to Samuel Foster.

Let us recall how we got to the Foster Circular dial (Fig 8): by bisecting the angle between the polar axis and the dial plane's axis. This is not the usual way of describing a Foster dial today, but I describe it this way for 2 reasons. The first is that it is the way Foster himself described the dial. The second is that it helps to make it clear that this beautiful device (Fig. 12) is in fact a Foster Circular dial.

This is the justly famous tercentenary sundial for the 300th anniversary of Greenwich Observatory.

I made the observation once on the Sundial Mailing List that this was simply a Foster Circular dial – and a well-known fellow dialist tried to correct me by insisting that the Foster dial must be horizontal. Of course, this is simply wrong. Read the text: Foster, *Elliptical or Azimuthal Horologiography*, pp.37-38 & pp.66-68. This is the dial's configuration (Fig. 13): the dial face has an inclination equal to the complement of the latitude, and the index itself is vertical. With a plane inclined this way, if we bisect the angle made by the polar axis and the axis of the dial face, what we get is exactly a vertical index – so we have a Foster Circular Dial. If that discussion is not enough to convince you that this is a Foster design, let me point out that Foster considers exactly this configuration (vertical index and inclined dial face) on pages 37 and 66 of his monumental book.

This dial deserves better treatment than it has received since its dedication. It was carted off to Cambridge when the Observatory offices were moved there. And then, when those offices closed, it went back to Herstmonceux Castle, which is no longer affiliated with the observatory. In fact, it is now owned by Queen's University (of Canada).

When the dial was originally unveiled at the anniversary celebration, there was talk of a new type of sundial – never-before seen –

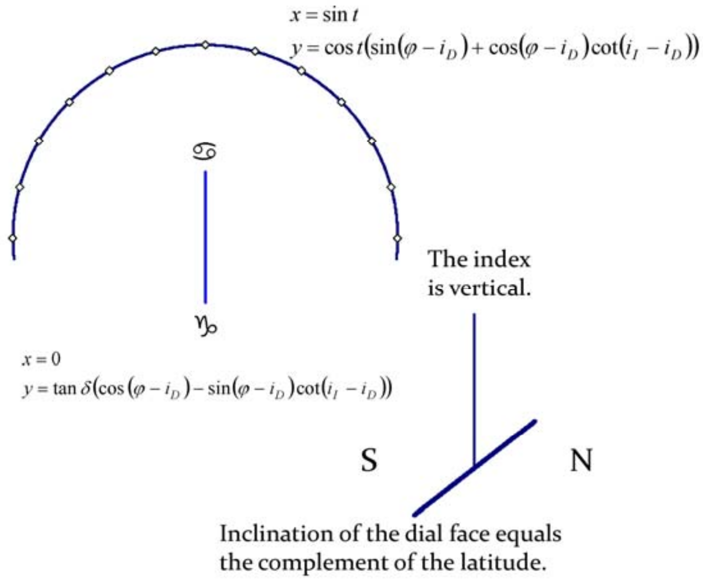


Fig. 13 Configuration of Tercentenary Dial

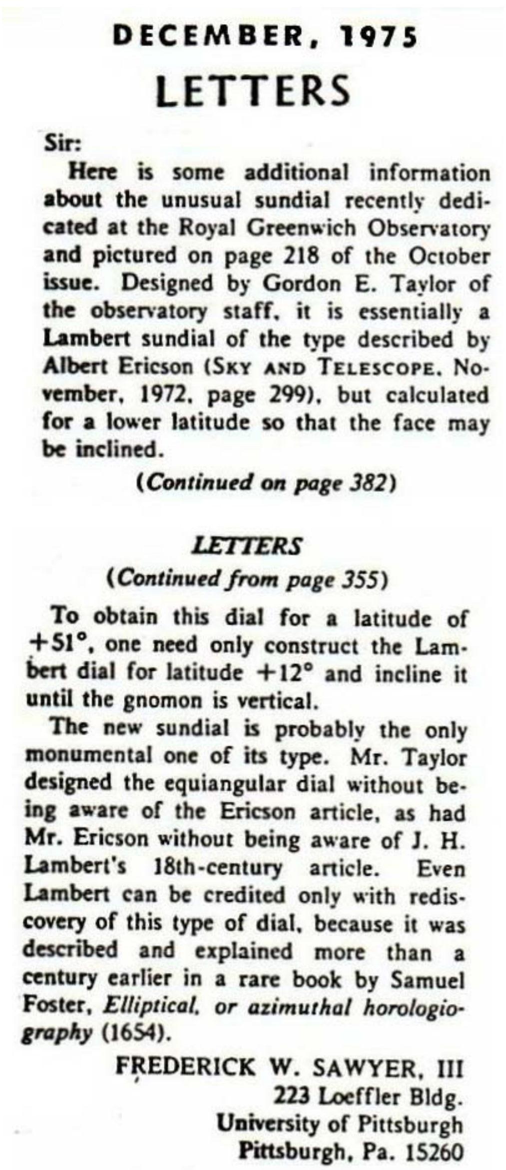
invented just for this occasion. I was in graduate school, and when my adviser returned from the ceremony with pictures of the mysterious new dial, all I could say was that it had first been invented in London two decades before the observatory was founded!

I wrote a letter to <u>*Sky & Telescope*</u>, which had published a brief article mentioning the new dial. They printed the letter in 1975 – and I thus had my first published item on sundials (Fig. 14).

I pointed out that the dial had been 'invented' earlier at least 3 times – including once in their own pages just 3 years earlier. I referred to it as a Lambert dial because I had not yet mounted my campaign to change the name to Foster/Lambert dial.

I also wrote a more extensive letter to the Observatory. It finally made its way to Gordon Taylor, the designer of the dial. He was kind enough to send a response, which unfortunately made it clear that that no one working on the dial had had any particular knowledge of dialing history. Taylor referred to 'a J. H. Lambert' – apparently never having heard of him. And their 'careful search' for earlier references to dials of this sort was restricted simply to Mayall, Cousins & Rohr. There is no mention in his reply of Foster – although he was the primary reason for my letter.

Taylor's article *Equiangular Sundials* (<u>Journal of the British Astronomical Association</u>, 86(1):7-17) appeared sometime later – and I was pleased to see that he had added a note while the article was in proof – pointing out that I had told him you could also make declining equiangular dials. His article had missed that point.



DECEMBER, 1975

LETTERS

Sir:

Here is some additional information about the unusual sundial recently dedicated at the Royal Greenwich Observatory and pictured on page 218 of the October issue. Designed by Gordon E. Taylor of the observatory staff, it is essentially a Lambert sundial of the type described by Albert Ericson (SKY AND TELESCOPE, November, 1972, page 299), but calculated for a lower latitude so that the face may be inclined.

(*Continued on page 382*)

LETTERS

(*Continued from page 355*)

To obtain this dial for a latitude of +51°, one need only construct the Lambert dial for latitude +12° and incline it until the gnomon is vertical.

The new sundial is probably the only monumental one of its type. Mr. Taylor designed the equiangular dial without being aware of the Ericson article, as had Mr. Ericson without being aware of J. H. Lambert's 18th-century article. Even Lambert can be credited only with rediscovery of this type of dial, because it was described and explained more than a century earlier in a rare book by Samuel Foster, *Elliptical, or azimuthal horologiography* (1654).

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Fig. 14 <u>*Sky & Telescope*</u> 50(6)

He acknowledged that I was right – but his explanation of how to do it has never been clear to me. All you really need to know is that the index must bisect the angle between the two axes. But unfortunately, Taylor never acknowledged in print anything having to do with Foster. Perhaps it just was not a good idea to acknowledge that an identical dial design had been published in London decades before the observatory was founded.

But let us now return to inclining the index. Suppose we go back to the vertical index and start inclining it to the south. The ellipse gets flatter, and the zodiac line gets longer (Fig. 15). Finally, when the index is pointing up into the equatorial circle, the hourlines all fall on a finite segment of a straight line (Fig. 16). This too makes for an interesting sundial.

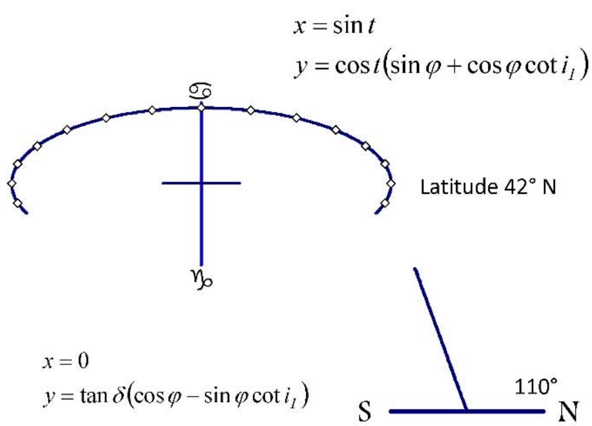

Fig. 15 Inclining the index to the South

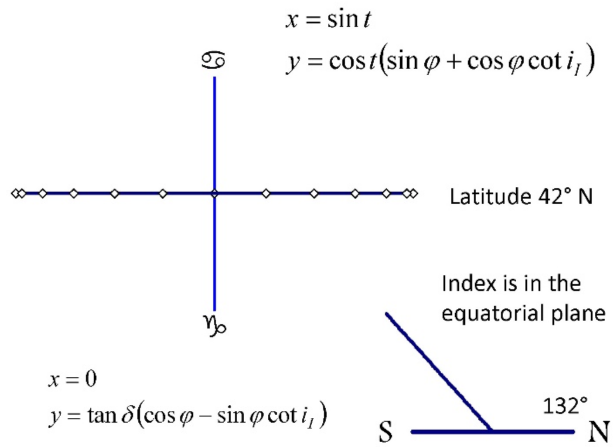

Fig. 16 Foster's Diametral Retrograde Dial

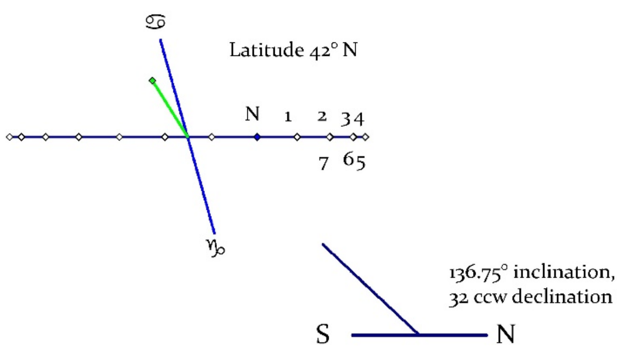

Fig. 17 Recreating Achaz' miracle of a retrograde shadow

The shadow goes retrograde – changing its direction twice a day – always remaining on this finite line segment. In fact, if we use a declining index (keeping it always in the plane of the equator), we can choose almost any time of day to force the shadow to change directions. We can recreate the biblical Miracle of Achaz any time we see fit.

In the case illustrated in Fig. 17, the shadow changes direction every day between 4 and 5pm. Foster covers this idea in the third section of his book.

But wait – there's more. Foster has more up his sleeve.

His dials can be drawn for an Index 'set any way' – inclining or declining. Begin, for example, with a Circular dial. And decline the index – rotate it around the vertical. Note that the subindicial and zodiac lines no longer coincide (Fig. 18). But we still have an elliptical sundial that works. We can make an elliptical dial for virtually any index we choose – no matter how it is situated.

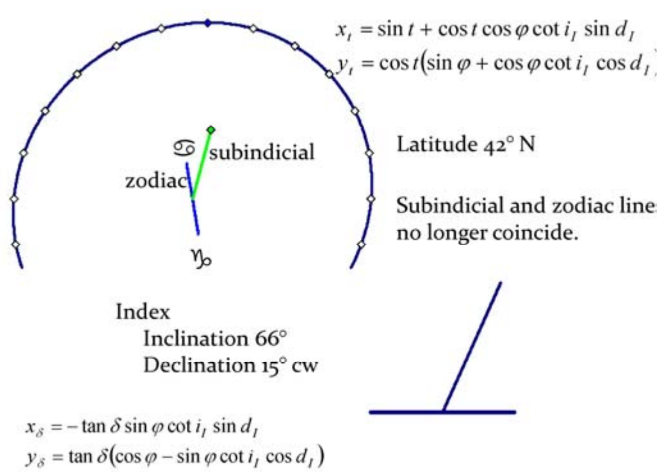

Fig. 18 Index Inclining and Declining

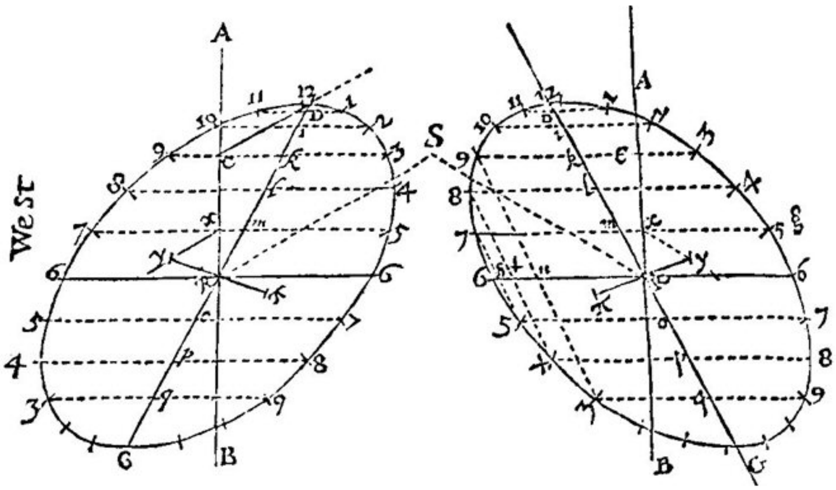

Fig. 19 Two dials with meridians not parallel to each other

In Fig. 19, we see Foster's drawings for two dials – with indices declining at different angles. *I have here set two of them, whose proper Meridians do not lie in one and the same right line, … but make angles one to the other. This is done, because … by that meanes they will alwayes set one another [i.e. form a self-orienting pair], which the Horizontal Diall with the single Ellipsis to an upright Index will not do at all times.* He correctly realizes that the traditional sundial in combination with the usual analemmatic forms a self-orienting pair that suffers the drawback that it can always be turned so that both dials of the pair indicate noon, no matter the actual time of day. With the two dials he shows here, there is no such drawback because the proper meridians of the dials are not parallel to each other – they can both indicate noon only at noon.

Craticles

At this point, we reach the end of Foster's book; however, we have not yet exhausted the surprises it contains. In at least three places in his book, Foster suggests that we can also make these dials in a *craticular* form; *e.g.* (p.38) *A double Diall depending upon this kinde, may be made to set it selfe, and to shew the houre in a Craticular forme...* (see also pp. 202, 204). Great! But what does that mean? 'Craticular' does not appear in the definitive Oxford English Dictionary, but we can get some of the meaning from the form of the word. It clearly alludes to something having the form of a craticle. And 'Craticle' does appear as a rare, obsolete word. Its earliest citation is given as 1657 – only slightly later than Foster's use. So the OED editors didn't actually read Foster either.

$$x_{♋} = \sin t + (\cos t \cos\varphi + \tan\delta \sin\varphi)\cot i_I \sin d_I$$

$$y_{♋} = \cos t \sin\varphi - \tan\delta\cos\varphi + (\cos t \cos\varphi + \tan\delta \sin\varphi)\cot i_I \cos d_I$$

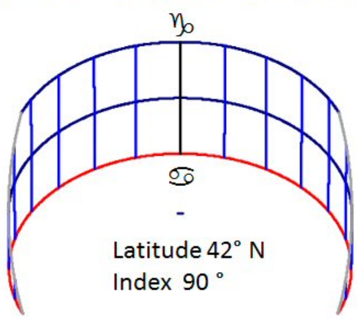

Fig. 20

A craticle is a latticework. So how do we make these dials with latticework? Keep the index stationary and draw multiple ellipses of hour points, each one corresponding to a different solar declination. If we do this with an analemmatic sundial, the ellipses and resulting hour lines appear as in Fig. 20. The index is stationary. The red ellipse is the summer solstice, and the other ellipses are for the equinoxes and winter solstice. The equations allow for an inclining and declining index.

In Fig. 21 we have a Foster circular dial in craticular form. But note that it is difficult to read times near

$$x_{♋} = \sin t + (\cos t \cos\varphi + \tan\delta \sin\varphi)\cot i_I \sin d_I$$

$$y_{♋} = \cos t \sin\varphi - \tan\delta\cos\varphi + (\cos t \cos\varphi + \tan\delta \sin\varphi)\cot i_I \cos d_I$$

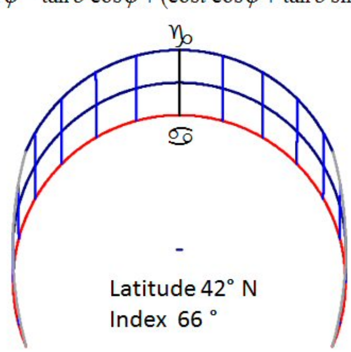

Fig. 21

r_0 *is the distance of the 6:00 point from the origin on the equinoxes.*

$$k = \sqrt{(\sin\varphi \cot i_I \sin d_I)^2 + (\cos\varphi - \sin\varphi \cot i_I \cos d_I)^2}$$

$$n = sign(\cos\varphi - \sin\varphi \cot i_I \cos d_I)$$

Then select a value V such that $|V| > k r_0 \tan 23.44^\circ$

And let $$r_\delta = \frac{r_0 V}{V - k n r_0 \tan\delta}$$

Fig. 22 Calculating more convenient radii

5, 6 or 7am or pm. because each ellipse is the same size. It would be nice to use a different radius for each solar declination - in such a way that the dial continues to work. How can we do this?

Here's the algebra (Fig. 22) that lets us go from a single radius for all ellipses to a unique one for each solar declination – and still keep the dials working. With that algebra done, we now have a general set of equations that produce better spacing among the hour and date lines.

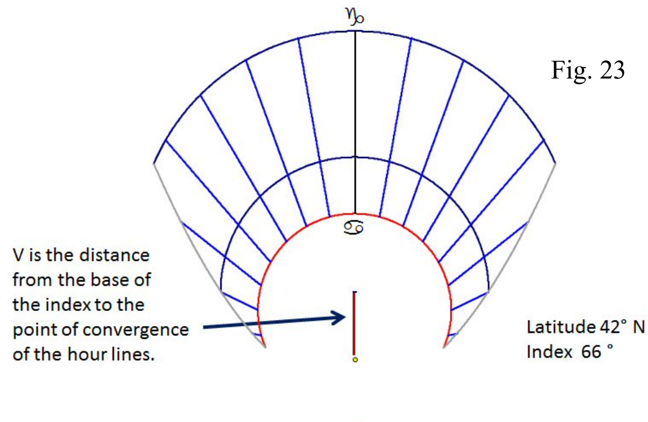

Fig. 23

$$x_{\delta} = r_\delta \sin t + r_\delta(\cos t \cos\varphi + \tan\delta \sin\varphi)\cot i_I \sin d_I$$
$$y_{\delta} = r_\delta(\cos t \sin\varphi - \tan\delta \cos\varphi) + r_\delta(\cos t \cos\varphi + \tan\delta \sin\varphi)\cot i_I \cos d_I$$

The V value we defined is the distance from the index to the point of convergence of the hour lines. By choosing different values for V, we can change the angles and whole appearance of the dial (Fig. 23).

So, for example, we can make the lines converge at a single point that is common to all the ellipses – or circles. This one arrangement has been discussed in recent years by Singleton, Sassenburg and Vercasson. Here (Fig. 24) we have an example by Michel Vercasson. (*Analemmatic Sundials With Fixed Styles*, The Compendium 17(1):8-11, Mar 2010).

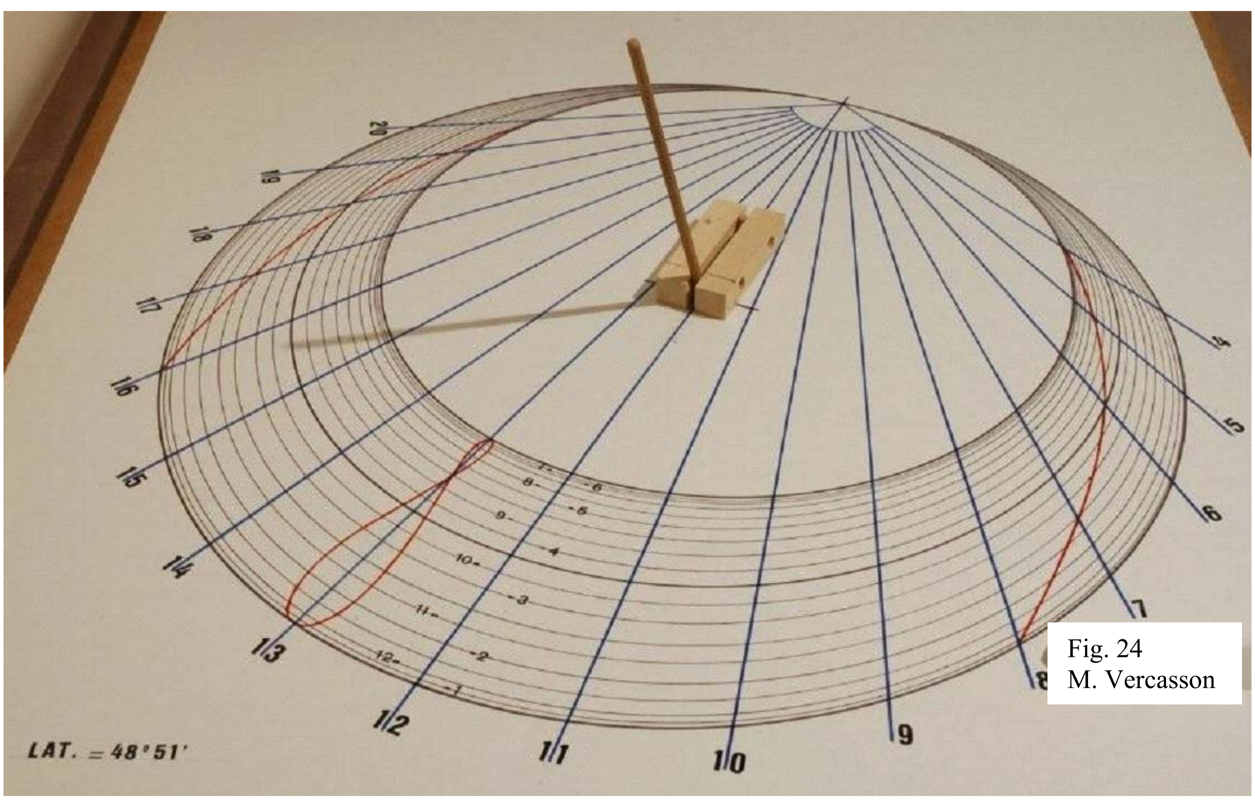

Fig. 24
M. Vercasson

But we can select 'almost' any point on the former zodiac line for convergence. In Fig. 25, the lines converge at a point north of the index. Note that the summer and winter solstices change places when the convergence is to the north.

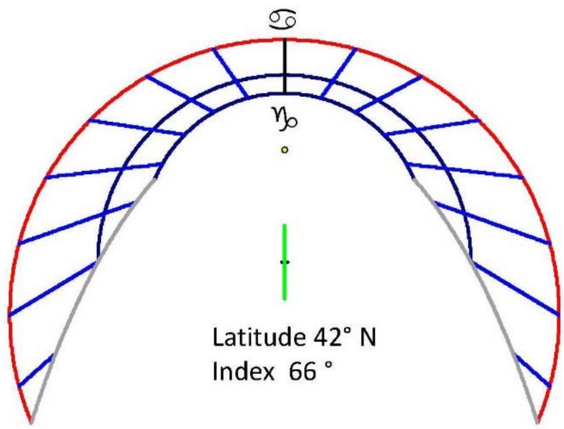

Fig. 25

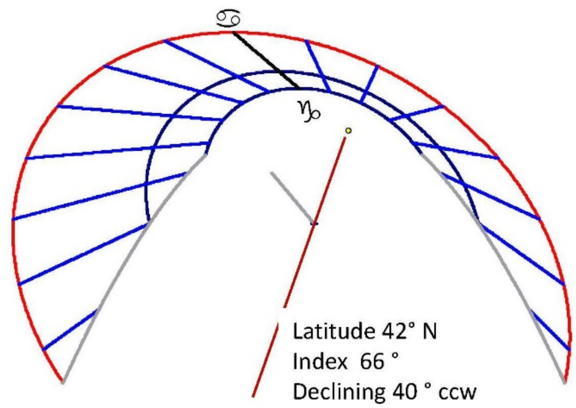

Fig. 26

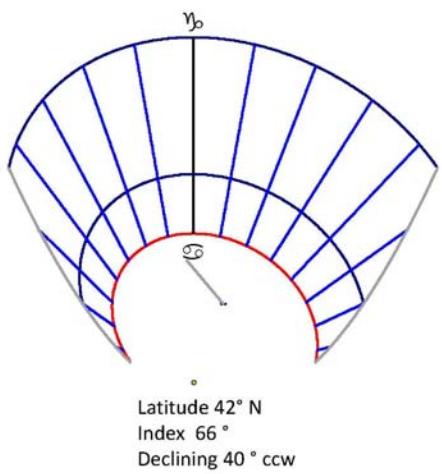

Fig. 27

If we decline the index, everything still works (Figs. 26 &27).

So by following up on the clues Foster left, we see that….

Given any plane and a stationary index pointing in any direction at any angle, it is possible to create a sundial using only ellipses and straight lines.

<u>Extending to a new dial</u>

Now I would like to extend the result even further – so let us ask a new question. Can we create sundials using a stationary index, straight lines, and only one ellipse? (Apart from the traditional polar axis dial, of course).

Let the single ellipse be the index itself. Start with an ellipse with an axis along the meridian line. Let the two semi-axes be *a* and *b* as shown in Fig 28.

Rotate the ellipse about the north/south axis so that it stands in the meridian plane. Fig. 29 shows the resulting craticle; this will work as a sundial, but it would not have satisfied Foster. These curves are not easily drawn without a modern computer.

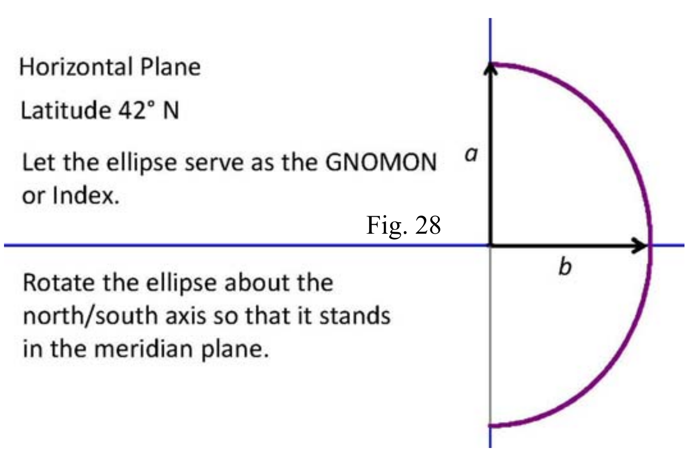

Fig. 28

But what happens if we look at other latitudes? The curves begin to straighten out as the latitude nears zero, and once we get to the equator we have straight lines (Fig 30). Here we show only the hours and 3 of the many declination lines.

In Figs. 31 and 32, we show the shadow of the elliptical index on the Summer Solstice at 3pm and on an Equinox at 10am, respectively. But we are only making use of shadows from the Southern part of the ellipse. Can we use the entire (half) ellipse? Yes, we can, with an extended craticle (Fig. 33).

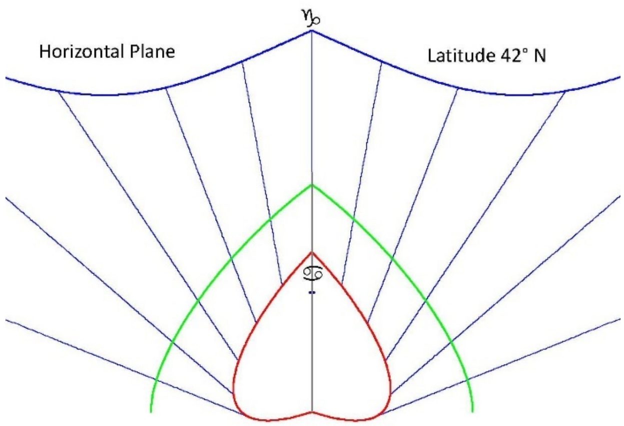

Fig. 29 Craticle at latitude 42 with elliptical gnomon

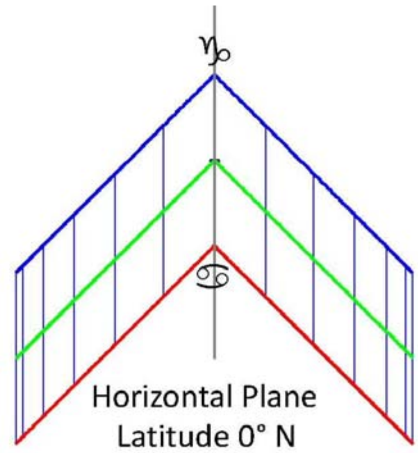

Fig. 30 Craticle at Equator with elliptical gnomon

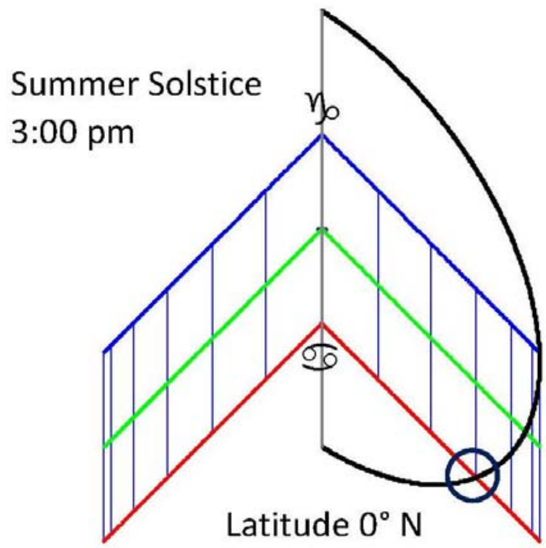

Fig. 31 Shadow at 3pm on Summer Solstice

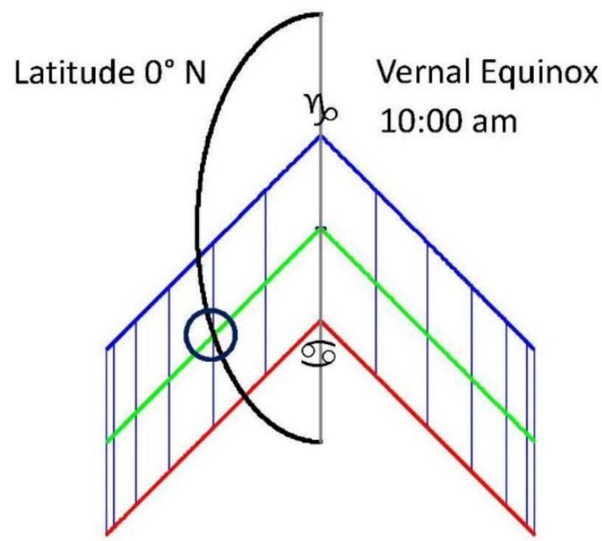

Fig. 32 Shadow at 10am on Vernal Equinox

Now note that we need not insist that this be a horizontal dial at the equator – it can equally well be a dial on any equivalent plane. This includes what we usually think of as a polar plane. Also a direct east-west vertical plane – and anything satisfying this equation $\tan\varphi = \tan i_D \cos d_D$ (where φ is the geographic latitude, i_D is the inclination of the dial, and d_D is its declination.

This is a new form of self-orienting polar sundial. If we look at the shadows on today's date, and turn the dial so that both readings show the same time – the dial will be correctly oriented (unless it's just turned to noon, or to a late afternoon time when we know it is actually morning!) As an example, see Fig. 34 which shows the gnomon's shadow at 4pm on the summer solstice; both readings match, so the dial is indeed oriented correctly.

The dial has a half-ellipse as gnomon, and the equations for the craticle are very easy: $x = b\sin t$ and $y = -b\tan\delta \mp a\sin t$.

Of course, the dial can also be built in a noncraticular way. Just create the proper single X of hour points (Fig. 35), and let it slide up and down on the meridian according to the current solar declination (Fig 36, showing summer solstice).

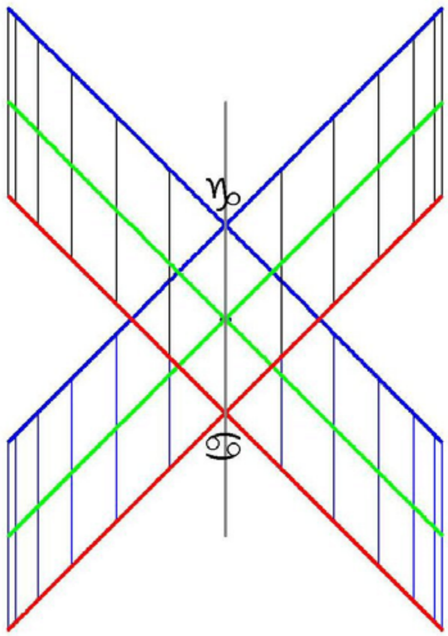

Fig. 33 Self-orienting, Double Craticle Polar Sundial with Elliptic Gnomon

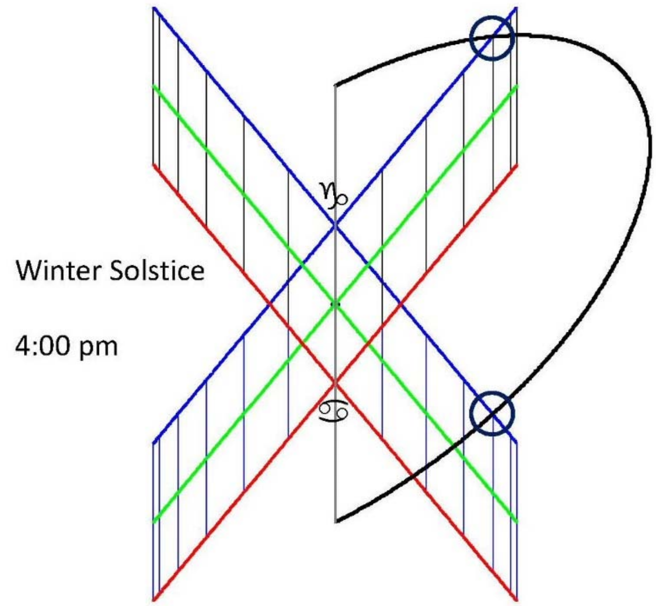

Fig. 34 Single shadow giving two matching times

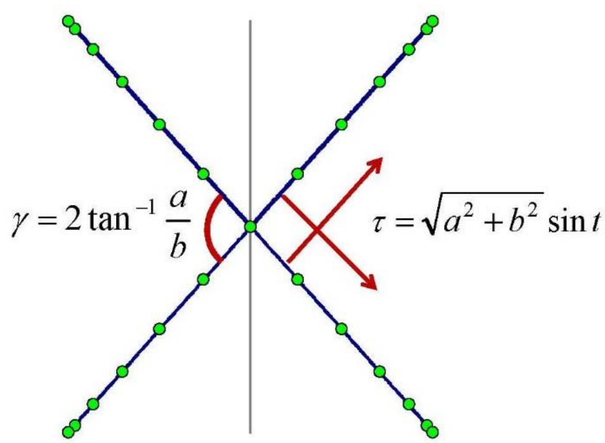

Fig 35 Equations for the noncraticular form

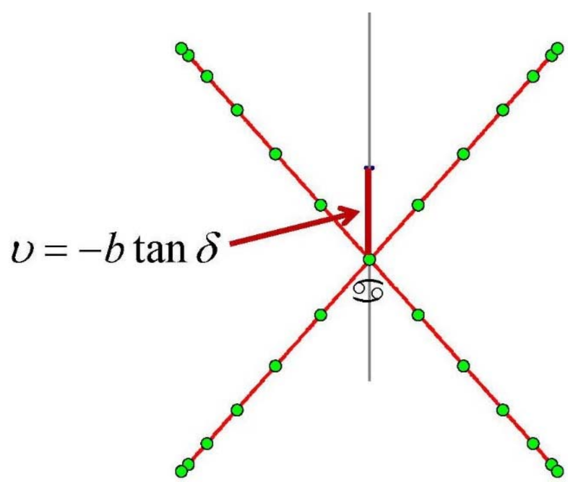

Fig 36 Equation for the declination adjustment form

Now, in closing, we can echo Samuel Foster:

> Hitherto we have had the whole business of Elliptical Horologiography, so far as that more cannot seem to be thought of, or required.

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