Compressed Gnomonic Sundials

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The purpose of this article is to demonstrate how a little-known map projection invented in the twentieth-century can be adapted to form the basis for new families of dials with interesting properties not seen in any other sundials.

The Map

The mapping we will consider was invented in 1914 by the German cartographer Hans Maurer, and then again independently in 1922 by Sir Charles Arden-Close in England. Scientific communication between countries in that era – particularly involving an invention that could have significant strategic importance – evidently was not particularly good. The projection has been known by a variety of names: compressed gnomonic, orthodromic, doubly azimuthal, 2-point azimuthal, Close-McCaw, Immier, *etc.* – each for a good specific reason. The projection does render all great circle paths as straight lines; in cartographic parlance, this feature makes it orthodromic. Another example with this feature is the gnomonic projection, which happens to be simply a special case of the one we will consider here. The projection is doubly azimuthal because it correctly renders azimuth angles measured from either of two focal points (whereas the usual azimuthal map projection has only one focal point); the angle a straight line path from one of these focal points to any other location on the map makes with the meridian line through the focal point always equals the azimuth of that location from the focal point. Finally, let me note that the projection can be obtained by tilting and then compressing a gnomonic projection along one axis – hence the name I prefer: compressed gnomonic.

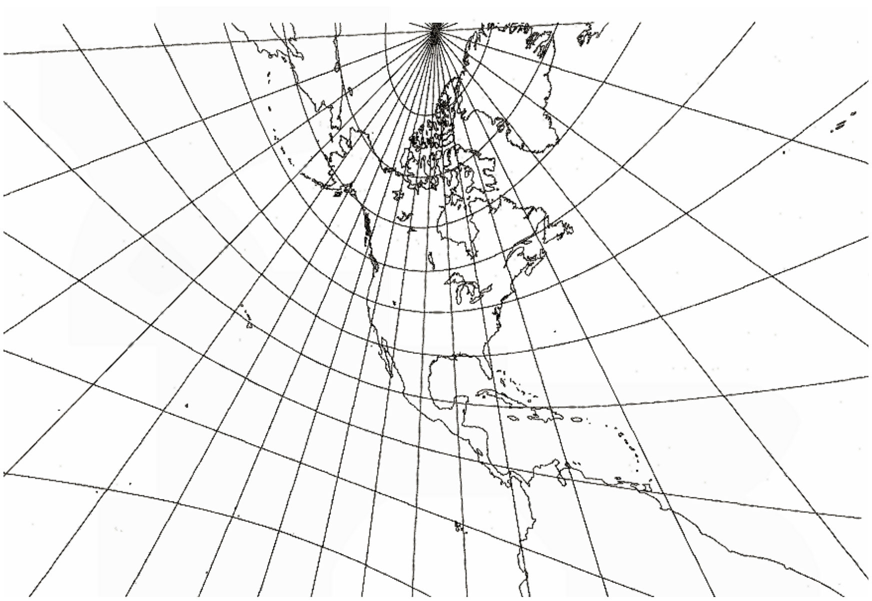

Fig. 1 Compressed Gnomonic map with foci at Washington & Honolulu

This projection was particularly useful for ships at sea in a pre-GPS world. Suppose such a ship has a compressed gnomonic map drawn so that there are radio receivers and transmitters positioned at its two focal points. If the ship transmits a radio message and these two receivers pick up the message with direction finding antennae, they can in turn transmit back to the ship its azimuth from each of the receivers. The ship's navigator can then simply draw lines at the given angles from the two focal points on the map, and the intersection of the two lines will be the ship's location.

Constructing the Projection

Before discussing the options this projection gives us for the design of sundials, let us first show how to construct it. We will leave a full mathematical treatment to an addendum that will accompany this issue of *The Compendium*. At this point, we will simply give instructions that will help the reader to follow the various steps in this projection.

1. Begin by selecting the two foci; for the sake of this example, let us consider two points on either side of the Atlantic: London (51°N 0°W) and a favorite camping site, conveniently located in Catskill State Park in New York (42°N 75°W).
2. Let d be half the length of the great circle arc joining these two sites. Let C be the point on this arc that is exactly midway between them. In our example, $d = 25.065°$ and C is 52.97°N 41.14°W.
3. Draw a standard gnomonic projection centered on the point C. This will render the great circle arc joining the foci (indeed – all great circle arcs) as a straight or orthodromic line.
4. Now rotate the projection so that the orthodromic line lies parallel to one of the axes. In our example, the projection must be rotated 12.22° clockwise for the great circle arc turned into an orthodromic line to lie parallel to the x-axis. (Of course, if we rotated it 77.78° counterclockwise, it would be parallel to the y-axis.)
5. Finally, compress the projection along the axis that is parallel to the line between the foci by multiplying all coordinates associated with that axis by cos d. In our example, each of the x-coordinates is multiplied by .905828.

Drawing the Dial

What results from these operations is a compressed gnomonic map with the two foci we identified at the outset. Since we are primarily interested in dialing rather than cartography, we can eliminate all the land masses and simply show the lines of latitude and longitude. We can replace the longitude lines with hour markings: 15° of longitude for each hour. And the only latitude lines that will be of interest are those between +23.5° and -23.5°; we can associate the latitude lines with the dates on which the sun's declination equals the given latitudes. What we end up with is shown in Figures 2 and 3. The longitude lines have been marked with equivalent Greenwich solar times, with noon being the meridian line passing through London. If Figure 2 is set horizontal at the London site with a vertical stile placed with its foot at the indicated site (where London would have been if we had retained the land masses!) and if the 12:00 Greenwich solar time line is aligned with the meridian as noted (South being indicated by 'S'), then the projection will serve as an azimuthal sundial.

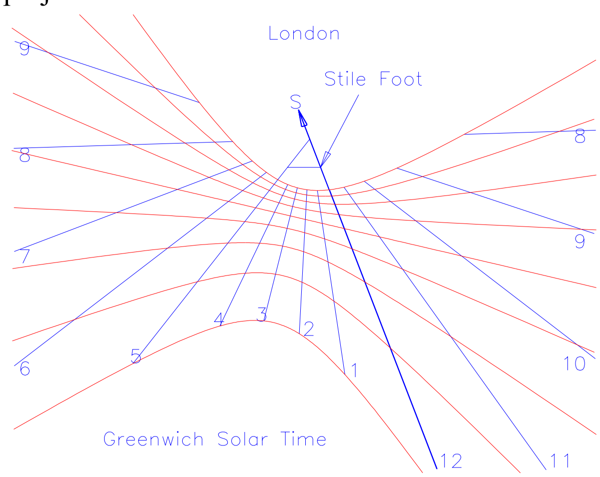

Figure 2

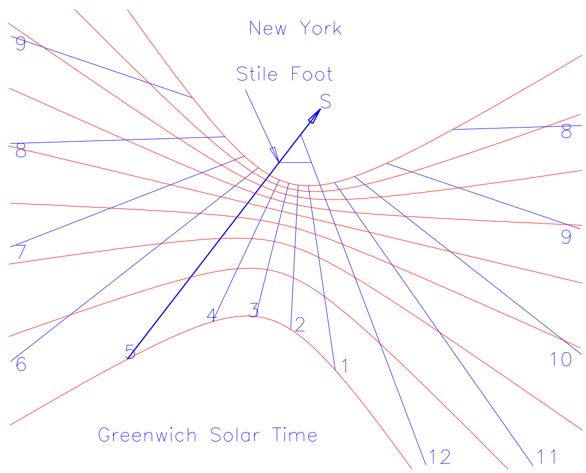

Figure 3

At any given moment, note where the stile's shadow falls on the curve corresponding to today's date. That point of intersection will tell you the current Greenwich solar time by its position among the hour lines. (Note: the day curves are not labeled here, but they run from summer solstice at the top to the winter solstice at the bottom, with the straight equinoctial line in the middle.)

Now, amazingly, if that exact same horizontal dial is transported to the camping site in New York, and if the vertical stile is set with its foot at the point corresponding to that site (see Figure 3) and the 5:00pm

Greenwich solar time line is aligned with the meridian, then the Greenwich solar time will again be given by the new stile's shadow.

This same horizontal azimuthal sundial works at two locations with nothing more than a re-placement of the vertical stile and a slight rotation of the dial for the new meridian – and those two locations are separated by 9° of latitude and 75° of longitude.

Of course, there is a slight problem with this dial. Note the distortion around, say, 11:00 am. On the winter solstice, the day curve does not intersect the hour line. This situation is alright for the New York site, since the sun is not even above the horizon there at 11:00 Greenwich solar time on the winter solstice. However, the sun is available in London at that date and time, but this sundial arrangement does not let us read the time because there is no intersection of the hour line and day curve.

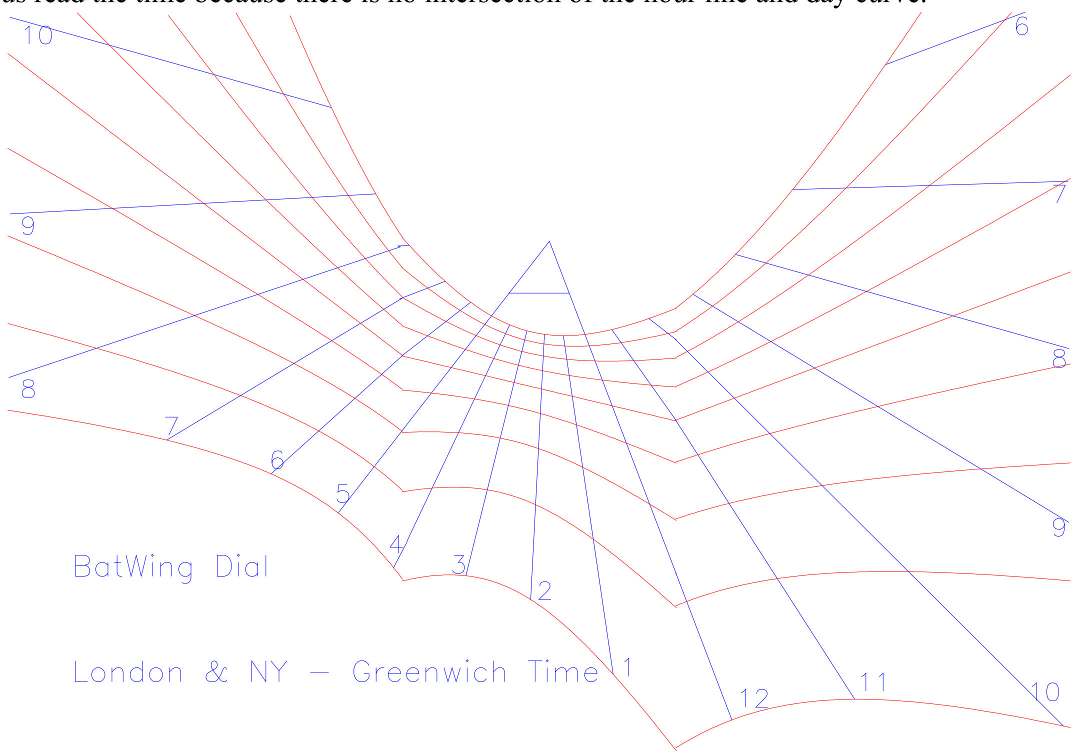

Figure 4

There are two ways to deal with this problem. The first is to weave different projections together to get something like the BatWing dial of Figure 4. This dial begins with a compressed gnomonic core for only those dates and times when the sun is above the horizon at both of the given locations. I then attach a standard gnomonic projection dial on the right-hand side for the other dates and times when the sun is up in London, and yet another standard gnomonic projection dial on the left-hand side for the dates and times when the sun shines only on New York. With a judicious choice of scale, these three projections can be stitched together and all three of them function correctly as azimuthal sundials whenever the appropriate stile casts a shadow on them.

A second, and perhaps preferable, method for dealing with the problem begins with rethinking the task we have set ourselves. In all probability, regardless of the site at which I am using the dial, I am more likely to be interested in local solar time than in Greenwich solar time. When it is noon in New York, I have no particular reason for wanting the dial to indicate that it is 5:00 pm Greenwich time. If we grant that this is a scenario more likely than the one with which I began, then it is possible to eliminate most, if not all, of the distortion that was introduced into the dial by basing it on a map whose foci were so far separated in longitude (*i.e.* in time). If we want the dial always to indicate local solar time, then we need only use one meridian line and we can base the dial on a map whose foci are separated only by latitude. Not only does this eliminate distortion in the dial, but it also significantly simplifies the equations needed for its construction. Let R be an arbitrary constant and let φ_a and φ_b be the two latitudes for which the dial is to be drawn. Then the foot of each stile is given by the points A and B, and the following parametric equations for x and y determine both the time lines and day curves. To draw a time line, set a value for t and find the points (x,y) that result from varying δ from +23.5° to -23.5°. Similarly, to obtain the points of a day curve, set a value for δ and then find the points (x,y) corresponding to the daylight range of values of t.

Equations for a Compressed Gnomonic Sundial with focal latitudes φ_a and φ_b and a single meridian

$$\varphi_0 = (\varphi_a + \varphi_b)/2 \qquad d = (\varphi_a - \varphi_b)/2 \qquad R = \text{an arbitrary constant}$$

$$A : (0, -R \sin d) \qquad B : (0, R \sin d)$$

$$x = R \frac{\cos\delta \sin t}{\cos\varphi_0 \cos\delta \cos t + \sin\varphi_0 \sin\delta} \qquad y = R\cos d \frac{\sin\varphi_0 \cos\delta \cos t - \cos\varphi_0 \sin\delta}{\cos\varphi_0 \cos\delta \cos t + \sin\varphi_0 \sin\delta}$$

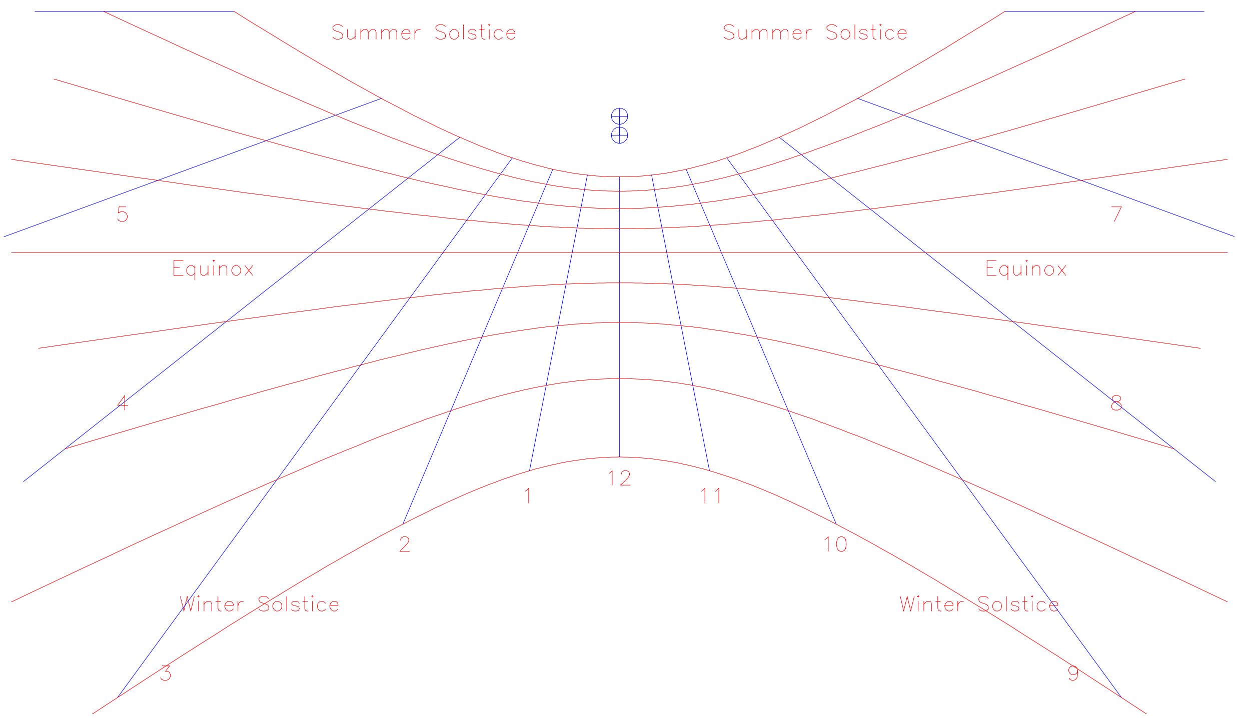

Figure 5.

In Figure 5 we see an example compressed gnomonic sundial for latitudes 51° and 42°. It appears to be not dissimilar from a standard gnomonic sundial – but it actually works quite differently. The standard gnomonic dial with a vertical rod gnomon or stile traces the day curve and points to the time with the end point of the gnomon's shadow, but for this dial, the end point of the shadow has no significance. All we are interested in is the direction of the shadow; we note where it intersects today's date and read the time from that point. The day curves themselves are reminiscent of the standard curves, but they are not the same. And of course the standard dial does not have two different points (noted here by two crossed circles) for the placement of two different vertical stiles, one for each of the two latitudes at which this new sundial can function as a horizontal azimuthal dial.

Further Developments

So, where else can this idea lead us? What we have so far is a new family of sundials that can be designed for two distinct latitudes – indeed, for two distinct latitude/longitude combinations. But how can we leverage this new capability?

Instead of a single dial for use at multiple latitudes, let us consider a dial intended for use at only one location – but in different possible orientations. So, for example, a vertical direct south dial at London (51°N) is equivalent to a horizontal dial at latitude 39°S. For our London flat, we can therefore construct a compressed gnomonic dial (Figure 6) that serves as a horizontal sundial, with stile foot at H, and as a vertical direct south dial, with stile foot at V. This dial can hang on the south wall of our patio in normal use, but when we are having lunch at our table in the sun and do not want to miss an important engagement, we can take the dial off the wall and place it on the table next to our cucumber sandwiches so that it will be readily in view throughout the meal.

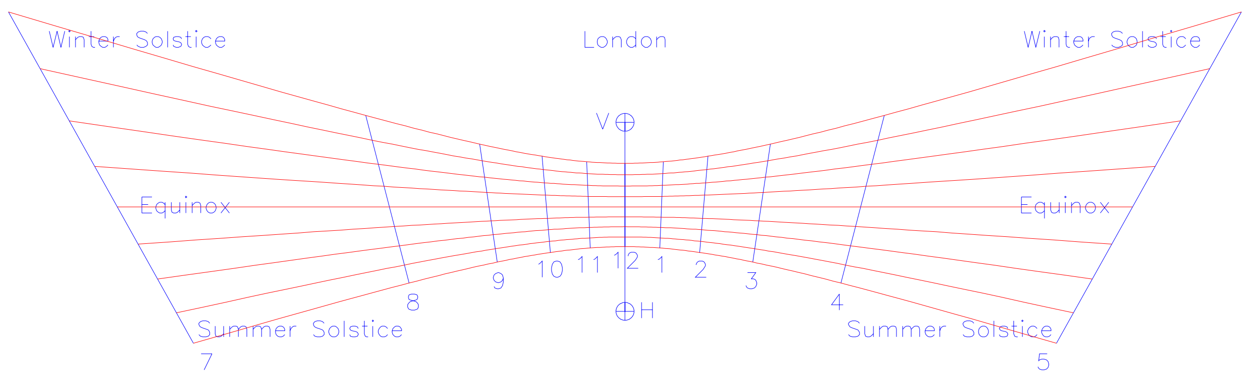

Figure 6

Technically speaking of course, in its vertical configuration this is not actually an azimuthal sundial – but the angle it measures is equivalent to the sun's azimuth at latitude 39°S and the time it indicates is perfectly correct for London.

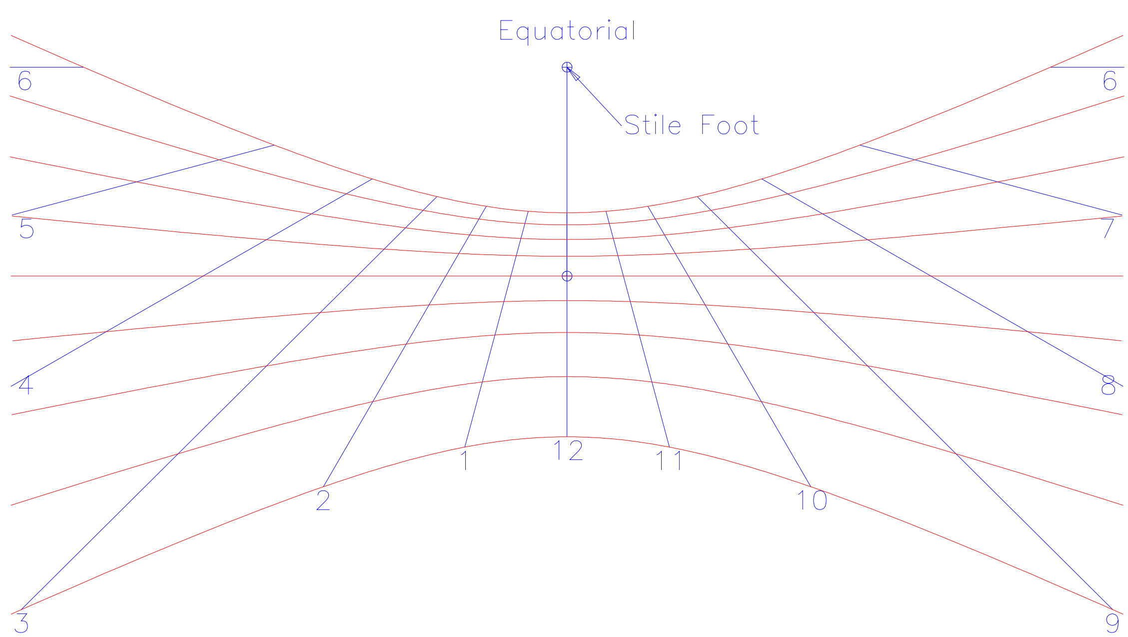

Figure 7

Let us take this idea a little further. In Figure 7 we have an equatorial compressed gnomonic dial (designed latitude 90°N, the location at which an equatorial dial for the spring/summer months would be horizontal). If the stile is placed as shown perpendicular to the dial face, and if the whole dial is inclined so that the stile is parallel to the celestial axis and the face lies in the equatorial plane, then this sundial functions as an equatorial. In fact, since the stile foot in this case is situated at the point of intersection of all the hour lines, we do not actually need to use the day curves to read the time – although they still function correctly whether we use them or not.

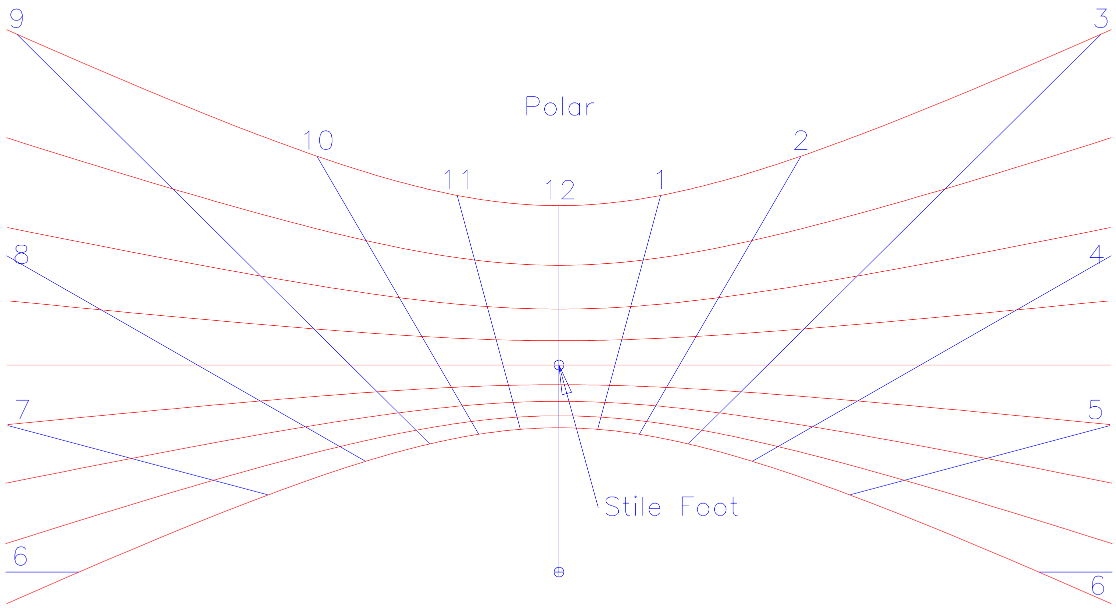

Figure 8

However, the sun shines on this face of the dial only during half the year. We can draw an analogous dial on the underside of this face and thus have a traditional equatorial dial (for which we would have needed none of the present analysis), or we can use this same dial face flipped around to be a polar dial – with its stile now moved to a new foot at the point of noon on the equinox line (Figure 8). Now the dial face is parallel to the celestial axis and the stile is in the equatorial plane. Interestingly, we now have a polar sundial with hour lines positioned at equal angles around noon.

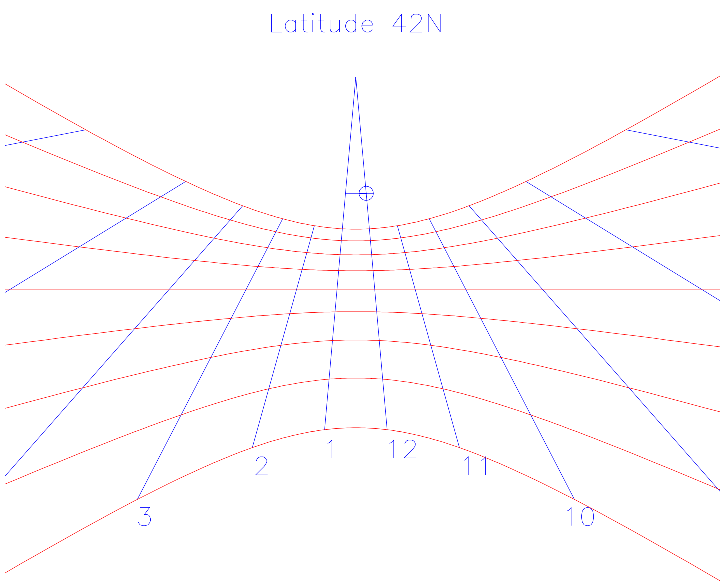

Figure 9

Taking Another Step

For the next step in this development, let us return to a feature we abandoned earlier: the possibility of a sundial with two meridian lines (*i.e.* with foci that differ in longitude as well as latitude). One way to minimize the distortion we saw in the earlier example is by keeping the longitude difference small.

Suppose we select the two focal locations to both be at 42°N latitude, separated by exactly 15° of longitude. The resulting dial will look like that in Figure 9. If we align the noon line on this horizontal dial to the local meridian at latitude 42°N and then place the vertical stile on the circled focal point, it's shadow will indicate the time throughout the year. If, however, we would like to make a change for Daylight Saving Time, we can simply realign the dial so that the 1:00pm hour line lies on the local meridian and reposition the stile so that it stands on the 1:00pm line (where it is intersected by the line which passes through the focus circle in Figure 9). With this simple reconfiguration, which amounted to simply rotating the dial through a small angle and moving the stile to a new location, the dial will now indicate local solar time plus one hour.

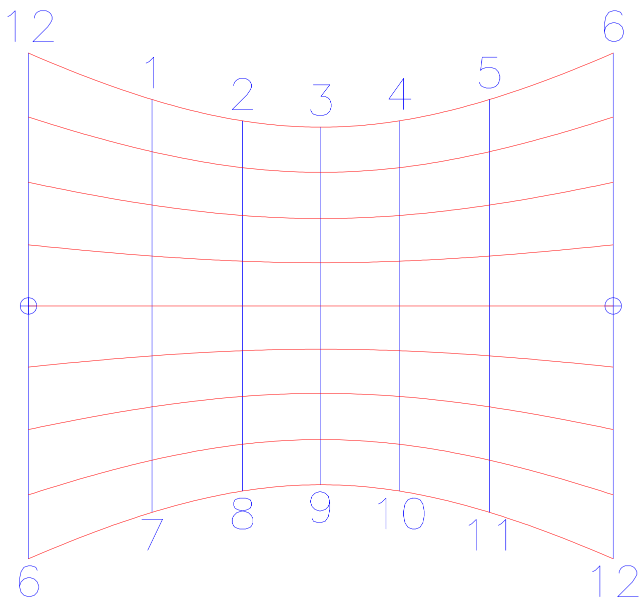

Figure 10

Suppose we now consider a polar sundial (Figure 10) with focal points on the equator and separated by 90° of longitude. We will only consider the portion of the dial in the space between the two foci. Because this dial is polar, it will receive the sun only on those dates and times in this interval – so we have a complete sundial with no appreciable distortion on the face. With this condensed gnomonic dial, we need not worry about the 6:00 hour lines being off at infinity – they fit very nicely into the dial face. In fact the equations for this dial are fairly simple:

$$x = \sin t / \left(\cos t + |\sin t|\right)$$

$$y = -\tan \delta / \left(\cos t + |\sin t|\right).$$

Place two stiles perpendicular to the dial face (at the circled points); the shadow of one of them will fall on the day curve for the current date and will there indicate the time.

This is not only a compact representation of the polar dial, but it works very nicely also as a vertical direct east or west dial if suspended at the appropriate angle on a wall. (Clockwise through an angle equal to the complement of the latitude for an east – morning – dial; counterclockwise for a west – afternoon – dial).

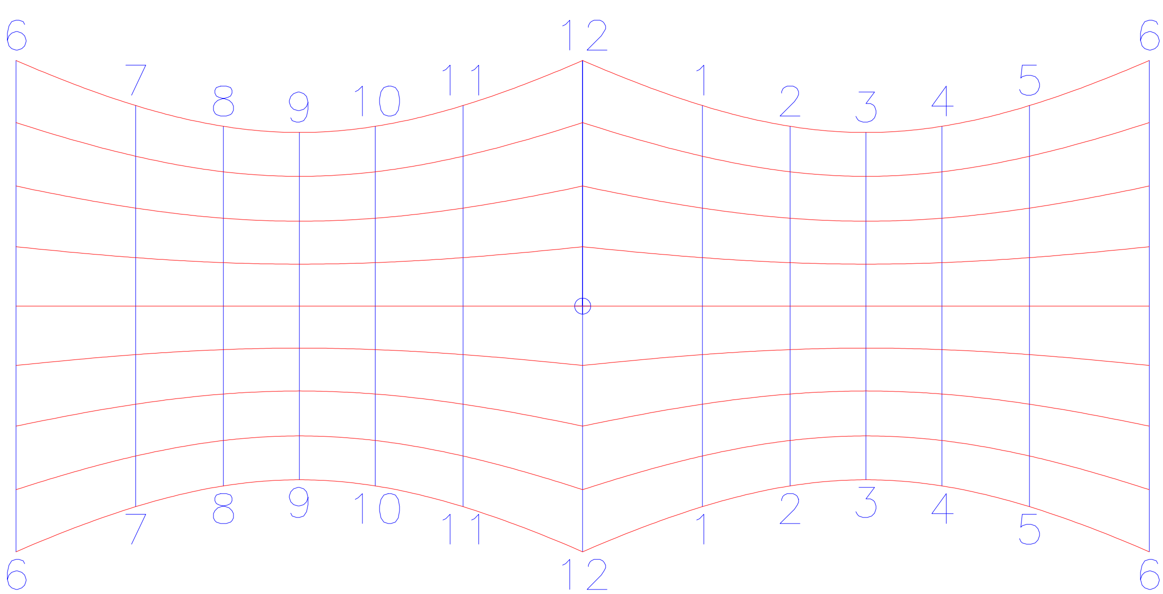

Figure 11

We can expand this dial (Figure 11) by again considering two foci on the equator separated by 90° of longitude, but this time use only the west focal point to anchor a stile. The dial will work fine for the afternoon. Now combine it with a similar dial using the same stile anchor as the eastern one of the pair of points. Stitching these two dials together yields a nicely symmetric polar sundial that actually involved three foci in its definition.

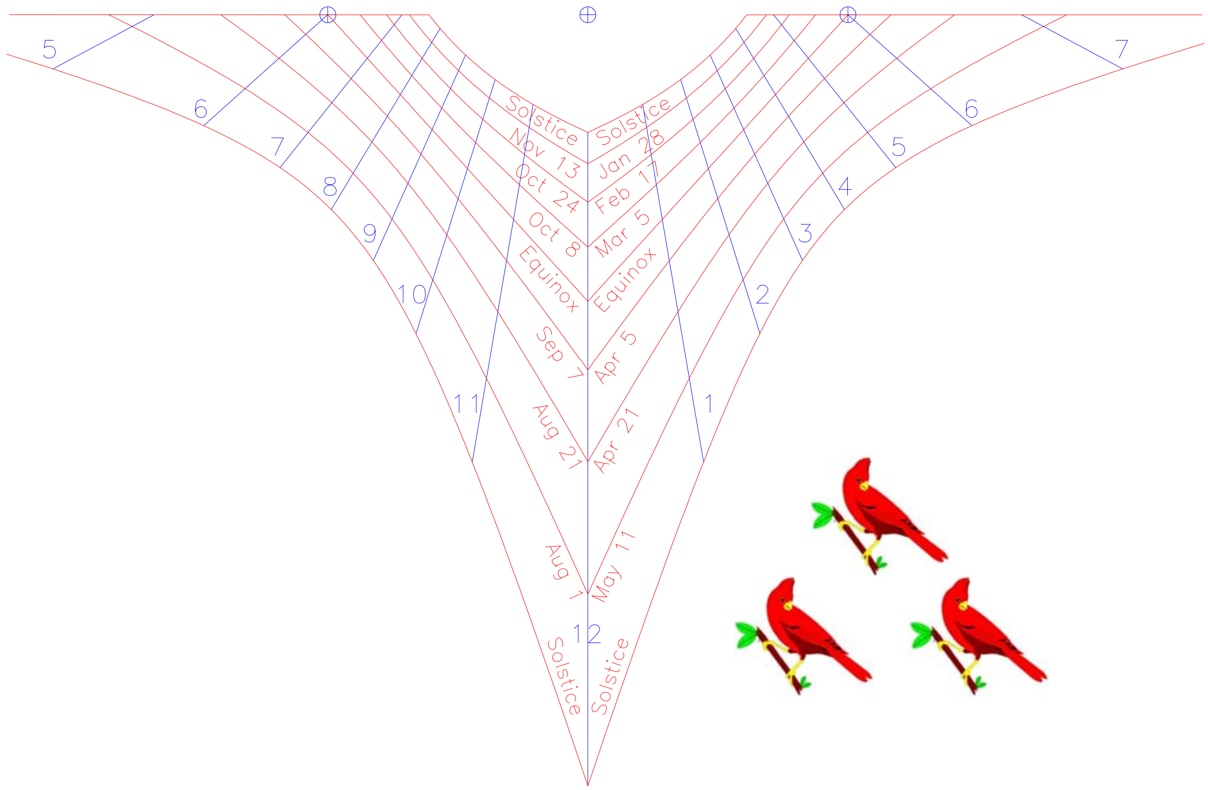

Figure 12. The Three Cardinals Sundial

This technique of stitching two dials together to avoid distortion can be used for yet another variation on compressed gnomonic dials. Let us select one focal point that is 90° south of our actual latitude and a second one that is on the equator 90° away from us. This combination results in a dial that works as a vertical direct south sundial for one of its stiles and as a vertical direct east dial for the other. It works nicely for the morning hours (represented by time lines falling between the two stiles); however the distortion is terrible in the afternoon. So we decide to discard the afternoon portion of the dial. We now consider a combination in which the second focal point is on the equator 90° to the other side of us. This produces a vertical direct south and vertical direct west dial that works well in the afternoon, but its morning portion is so distorted that it clearly should be discarded. Fine. Stitch these two remaining halves of dials together at their noon lines and we have the Three Cardinals Sundial (Figure 12). With perpendicular stiles at the three indicated points, the dial works when hung on a wall facing any one of the three cardinal directions of east, south or west. There is no need to tilt the dial (as was the case with the previous polar dial used as a direct east/west); simply hang it on a nail and read the time.

In closing, I will point out that we have not yet exhausted all possibilities. It is also possible to construct a compressed gnomonic dial designed for a single location and a single orientation but which uses both foci to indicate the time. The resulting dial, easily adjustable for the equation of time, will be the subject of an article in the next issue of *The Compendium*.

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Compressed Gnomonic (Two-Point Azimuthal or Orthodromic) Projection

Foci: (φ_A, λ_A) (φ_B, λ_B) [latitude (N is positive) and longitude (W is positive)]

Find the point (φ_0, λ_0) midway on the great circle arc between A and B. This will be the center of the projection.

$$Az_1 = \tan^{-1}\left[\frac{\cos\varphi_B \sin(\lambda_A - \lambda_B)}{\cos\varphi_A \sin\varphi_B - \sin\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B)}\right] \qquad d = \frac{\cos^{-1}[\sin\varphi_A \sin\varphi_B + \cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B)]}{2}$$ = arclength/2

Center: (φ_0, λ_0)

$$\varphi_0 = \sin^{-1}(\sin\varphi_A \cos d + \cos\varphi_A \sin d \cos Az_1) \qquad \lambda_0 = \lambda_A - \tan^{-1}\left[\frac{\sin d \sin Az_1}{\cos\varphi_A \cos d - \sin\varphi_A \sin d \cos Az_1}\right]$$

For specific points, find the usual gnomonic projection coordinates (x_g, y_g) with (φ_0, λ_0) as center.

Gnomonic projection:

$$\cos z = \sin\varphi_0 \sin\varphi + \cos\varphi_0 \cos\varphi \cos(\lambda_0 - \lambda) > 0$$

$$x_g = R\cos\varphi \sin(\lambda_0 - \lambda)/\cos z \qquad y_g = R[\cos\varphi_0 \sin\varphi - \sin\varphi_0 \cos\varphi \cos(\lambda_0 - \lambda)]/\cos z$$

The azimuths from the center to the focus points are Az_A and Az_B:

$$Az_A = \tan^{-1}\left[\frac{\cos\varphi_A \sin(\lambda_0 - \lambda_A)}{\cos\varphi_0 \sin\varphi_A - \sin\varphi_0 \cos\varphi_A \cos(\lambda_0 - \lambda_A)}\right] \qquad Az_B = \tan^{-1}\left[\frac{\cos\varphi_B \sin(\lambda_0 - \lambda_B)}{\cos\varphi_0 \sin\varphi_B - \sin\varphi_0 \cos\varphi_B \cos(\lambda_0 - \lambda_B)}\right]$$

Rotate the projection about the center to place the focus points on one of the axes. Compress this axis by multiplying it by $\cos d$.

Rotation and compression:

$$Az = \begin{cases} Az_A, & \text{if A is east of B} \\ Az_B, & \text{otherwise} \end{cases}$$

(To x-axis)
$$x = \cos d(x_g \sin Az + y_g \cos Az)$$
$$y = y_g \sin Az - x_g \cos Az$$

(To y-axis)
$$x = x_g \cos Az - y_g \sin Az$$
$$y = \cos d(y_g \cos Az + x_g \sin Az)$$

Equations for a Compressed Gnomonic Sundial with focal latitudes φ_a and φ_b and a single meridian

$\varphi_0 = (\varphi_a + \varphi_b)/2 \qquad d = (\varphi_a - \varphi_b)/2 \qquad R$ = an arbitrary constant $\qquad A:(0, -R\sin d) \qquad B:(0, R\sin d)$

$$x = R\frac{\cos\delta\sin t}{\cos\varphi_0\cos\delta\cos t + \sin\varphi_0\sin\delta} \qquad y = R\cos d\frac{\sin\varphi_0\cos\delta\cos t - \cos\varphi_0\sin\delta}{\cos\varphi_0\cos\delta\cos t + \sin\varphi_0\sin\delta}$$

Demonstration:

$$\cot Ang_a = \frac{y + R\sin d}{x}$$
$$= \frac{\cos d\sin\varphi_0\cos\delta\cos t - \cos d\cos\varphi_0\sin\delta + \sin d\cos\varphi_0\cos\delta\cos t + \sin d\sin\varphi_0\sin\delta}{\cos\delta\sin t}$$
$$= \frac{(\cos d\sin\varphi_0 + \sin d\cos\varphi_0)\cos t - (\cos d\cos\varphi_0 - \sin d\sin\varphi_0)\tan\delta}{\sin t}$$
$$= \frac{\sin(\varphi_0 + d)\cos t - \cos(\varphi_0 + d)\tan\delta}{\sin t}$$
$$= \frac{\sin\varphi_a\cos t - \cos\varphi_a\tan\delta}{\sin t} = \cot(\textit{Solar azimuth at latitude A, solar declination } \delta \textit{ and time } t)$$

Ang_a is the angle between the meridian line and the line from the foot of Stile A to the intersection of day curve δ and time line t.

$$\cot Ang_b = \frac{y - R\sin d}{x}$$
$$= \frac{\cos d\sin\varphi_0\cos\delta\cos t - \cos d\cos\varphi_0\sin\delta - \sin d\cos\varphi_0\cos\delta\cos t - \sin d\sin\varphi_0\sin\delta}{\cos\delta\sin t}$$
$$= \frac{(\cos d\sin\varphi_0 - \sin d\cos\varphi_0)\cos t - (\cos d\cos\varphi_0 + \sin d\sin\varphi_0)\tan\delta}{\sin t}$$
$$= \frac{\sin(\varphi_0 - d)\cos t - \cos(\varphi_0 - d)\tan\delta}{\sin t}$$
$$= \frac{\sin\varphi_b\cos t - \cos\varphi_b\tan\delta}{\sin t} = \cot(\textit{Solar azimuth at Latitude B, solar declination } \delta \textit{ and time } t)$$

Ang_b is the angle between the meridian line and the line from the foot of Stile B to the intersection of day curve δ and time line t.